The principles of mathematics and the problem of sets

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In its issue of the 30 March 1905, the *Revue* points out certain contradictions that we encounter in the general set theory.

It is not necessary to go so far as the theory of ordinal numbers to find such contradictions. Here is one that presents itself the moment we study the continuum and to which some others could probably be reduced.

I am going to define a certain set of numbers, which I will call the set E, through the following considerations.

Let us write all permutations of the twenty-six letters of the French alphabet taken two at a time, putting these permutations in alphabetical order; then, after them, all permutations taken three at a time, in alphabetical order, then, after them all permutations taken four at a time, and so fourth. These arrangements may contain the same letter repeated several times; they are permutations with repetition.

For any integer p, any permutation of the twenty-six letters taken p at a time will be in a table; and, since everything can be written with finitely many words is a permutation of letters, everything that can be written will be in a table formed as we have indicated.

The definition of a number being made up of words, and these with letters, some of these arrangements will be definitions of numbers. Let us cross out our permutations all those which are not definitions of numbers.

Let u_1 at be the first number defined by a permutation, u_2 the second, u_3 the third, and so on.

We thus have, written in a definite order, *all the numbers defined using a finite number of words.*

Therefore, the numbers that can be defined by finitely many words form a denumerable infinite set.

Now here comes the contradiction. We can form a number not belonging to this set. "Let p, be the digit in the nth decimal place of the nth number of the set E; let us form a number having 0 for the integral part and, in its nth decimal place, $p + 1$ if p is not 8 or 9, and 1 otherwise." This number N does not belong to the set E. If it were the nth number of the set E, the digit in its n decimal place would be in the nth decimal place of that number, which is not the case.

I denote by G the collection of letters between quotation marks.

The number N is defined by the words of the collection G, that is, by finitely many words; hence it should belong to the set E. But we have seen that it does not.

Such is the contradiction.

Let us show that this contradiction is only apparent. We come back to our permutation. The collection G of letters is one of these permutations; it will appear in my table. But, at the place it occupies, it has no meaning. It mentions the set E, which has not been yet defined. Hence I have to cross it out. The collection G has meaning only if the set E is totally defined, and this is not done except by infinitely many words. *Therefore there is no contradiction.*

We can make a further remark. The set containing [the elements of] the set E and the number N represents a new set. This set is denumerably infinite. The number N can be inserted into the set E at a certain rank k if we increase by 1 the rank of each number of rank [equal to or] greater than k. Let us still denote E the thus modified set. The collection of words G will define a number N' *distinct from* N, since the number N occupies rank k and the digit in the kth decimal place of N' is not equal to the digit in the kth decimal place of the kth number of set E.